1. Exercises from 5.2

-Remind them of the definition of line integrals -Show them Green's theorem in plane

PROBLEM 1. *(Folland 5.2.1(a,c))*

Part a): Let C be the unit circle, traversed *clockwise*. Last time, we computed the following line integral by hand

$$\int_C (x-y)dx + (x+y)dy = -2\pi$$

We now do the same computation using Green's theorem. Notice that the orientation reversal of the boundary introduces a minus sign in the final answer.

$$\int_C (x-y)dx + (x+y)dy = -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = -\iint_D (1-(-1))\, dA = -2\iint_D dA = -2\pi$$

Which agrees with the computation from last week. In the last step, we simply used that the area of the unit disk is π.

Part c): Compute:

$$\int_C (x^2+10xy+y^2)dx + (5x^2+5xy)dy$$

Where C is the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$ oriented counterclockwise.
Again, we apply Green's theorem.

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(5x^2+5xy) = 10x+5y$$
$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(x^2+10xy+y^2) = 10x+2y$$

So the integral is:

$$\int_C (x^2+10xy+y^2)dx + (5x^2+5xy)dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_0^2\int_0^2 3y\, dx\, dy = 12$$

PROBLEM 2. *(Folland 5.2.2) Compute, both directly and using Green's theorem,*

$$\int_{\partial S} -x^2y dx + xy^2 dy$$

When S is the region $1 \le x^2+y^2 \le 4$

The boundary is given by two circles, one having radius 1 and the other having radius 2. The orientation of the inner circle is clockwise, while the orientation of the outer circle is counter clockwise.

$$\gamma_r(t) = (r\cos t, r\sin t) \Rightarrow \gamma_r'(t) = (-r\sin t, r\cos t)$$

So

$$\mathbf{F}(\gamma_r(t))\cdot\gamma_r'(t) = \begin{pmatrix} -r^3\cos^2 t\sin t \\ r^3\cos t\sin^2 t\end{pmatrix}\cdot\begin{pmatrix} -r\sin t \\ r\cos t\end{pmatrix} = 2r^4\sin^2 t\cos^2 t$$

Now the line integral of $\mathbf{F}$ along a circle of radius r, C_r, can be computed:

$$\int_{C_r} F_1 dx + F_2 dy = \int_0^{2\pi} \mathbf{F}(\gamma_r(t))\cdot\gamma_r'(t)\, dt = \int_0^{2\pi} 2r^4\sin^2 t\cos^2 t\, dt$$

Applying the power reduction trigonometric formulas:

$$\int_0^{2\pi} 2r^4\sin^2 t\cos^2 t\, dt = \frac{2r^4}{4}\int_0^{2\pi}(1-\cos 2t)(1+\cos 2t)\, dt = \frac{2r^4}{4}\left(2\pi - \int_0^{2\pi}\cos^2 2t\, dt\right) = \frac{\pi r^4}{2}$$

Now we can evaluate the line integral:

$$\int_{\partial S} F_1 dx + F_2 dy = \int_{C_2} F_1 dx + F_2 dy - \int_{C_1} F_1 dx + F_2 dy = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$

The problem is much simpler when we apply Green's theorem:

$$\iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_S y^2 - (-x^2)\, dA = \int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta = 2\pi \left[\frac{r^4}{4} \right] \Bigg|_1^2 = \frac{15\pi}{2}$$